→ Basic Random Processes

→ N kicks: $P_N \to \dfrac{\exp[-r^2/2N\langle r^2\rangle]}{[\langle r^2\rangle N]^{d/2}}$

CLT

$\langle r^2 \rangle \to$ ms kick

→ how show?

→ Transport – Basic Ideas – Diffusion

- previously developed theory of fluctuations near equilibrium

- now, → $\boxed{\text{transport}}$ / relaxation

- prblm. is one of coulomb scattering in stable plasma (i.e. resistivity)

  ⇒ weak deflection

$$\boxed{\Delta p_\perp \sim \int_{-\infty}^{+\infty} dt \frac{e^2}{r^2}\frac{\rho}{r}}$$

∴

- Approach via:
  - general theory of pdf evolution via series of small, random kicks. → weak deflection
    - ⇒ Diffusion, Central Limit Thm.
  - Fokker-Planck Eqn. (Generic)
  - Lenard-Balescu Eqn. (extends test particle model)
  - FPE + Rosenbluth Potentials

→ Basics:

Can discretize process (N jumps) with probability of jump by $\underline{r}$ in step

Chapman-Kolmogorov

1 step back (no memory)

$$P_{N+1}(\underline{X}) = \int \underbrace{p(\underline{r})}_{\text{step pdf}}\, P_N(\underline{X}-\underline{r})\, d\underline{r}$$

$\underline{X}$ = position, cumulative

$\underline{r}$ = jump

For small steps, expand to 2nd order: (why)

$$P_{N+1}(\underline{X}) = \int d\underline{r} \left\{ P_N(\underline{X}) - \underline{r}\cdot\underline{\nabla} P_N(\underline{X}) + \frac{1}{2}\, \underline{r}\cdot\underline{\nabla}\,\underline{\nabla} P_N \cdot \underline{r} \right\} p(\underline{r})\, d^d\underline{r}$$

$$\cong P_N(\underline{X}) + \frac{\langle \underline{r}\cdot\underline{r}\rangle}{2d} \nabla^2 P_N(\underline{X})$$

no bias (drift)

→ normalizable p, $\langle r^2 p\rangle$ exists.

→ no bias

→ thus, have:

$d^d r = dr$

$= r^{d-1} dr\, d\underline{\Omega}$

$$\frac{P_{N+1}(\underline{X}) - P_N(\underline{X})}{\Delta t} = \frac{\langle r^2\rangle}{2d\,\Delta t} \nabla^2 P_N$$

diffusion equation!

$$\boxed{\begin{aligned} \frac{\partial P}{\partial t} &= D \nabla^2 P \\ D &= \langle r^2\rangle / 2d\,\Delta t \end{aligned}}$$

$P \to \rho$

$\rho(\underline{x}, 0) = \delta(\underline{x})$

$$\frac{\partial \rho}{\partial t} = -k^2 D \rho$$

so $\frac{\partial \rho}{\partial t} = -Dk^2 \rho$

FT

$$\rho(k, t) = e^{-Dk^2 t} \rho(k, 0)$$
$$= e^{-Dk^2 t} \quad 1.$$

so Inverse $\Rightarrow$

$$\boxed{\rho(\underline{x}, t) = \frac{e^{-x^2/4Dt}}{(4\pi D t)^{d/2}}} \to \text{pdf for position, in time}$$

i.e. $\rho(\underline{k}, t) = e^{-Dk^2 t}$
$$= e^{-D(k_x^2 + k_y^2 + k_z^2)t}$$

so

$$\rho(\underline{x}, t) = \int e^{i(k_1 x_1 + k_2 x_2 + k_3 x_3 + \dots)} e^{-D(k_1^2 + k_2^2 + \dots k_d^2)t} \, dk_1 \, dk_2 \dots dk_d$$

= above.

$\therefore$

$$\boxed{\rho(\underline{x}, t) \sim \frac{1}{t^{-1/2}} e^{-x^2/4Dt}}$$

return
1D
2D
3D

$x^2 = \underline{x} \cdot \underline{x}$

Then, re-discretizing:

$$P_N(\underline{x}) = \frac{\exp\left[-dx^2/2\langle r^2\rangle N\right]}{(2\pi\langle r^2\rangle N/d)^{d/2}}$$

N steps

$$\boxed{P_N(x) \sim \frac{e^{-d x^2/2\langle r^2\rangle N}}{\left(\langle r^2\rangle N\right)^{d/2}}}$$

$P_N(x)$:

- converges, time asymptotically, to Gaussian, with width
  $\sim N^{1/2}\langle r^2\rangle^{1/2} \sim N^{1/2} r_{rms}$
- $\sim N^{-d/2}$.

$\Rightarrow$ quickie demo of Central Limit Theorem.

C.L.T.:

Let $x_1, x_2, \ldots$ be a sequence of:

- independent
- identically distributed

random variables, each with mean μ and variance σ^2

i.e. $\overline{x_i} = \mu$

$$\langle (x_i - \overline{x_i})^2 \rangle = \sigma^2 .$$

then the distribution of:

$$\frac{x_1 + x_2 + \cdots x_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{n\to\infty} \text{normal (Gaussian)}$$

i.e.

$$P\left\{ \frac{x_1 + x_2 + \cdots x_n - n\mu}{\sigma\sqrt{n}} \leq \varepsilon \right\} = \int_{-\infty}^{\varepsilon} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx$$

Note: Holds for all sequences

- independent → no correlations
- identically distributed
  i.e. no 'special' step, → intermittency.
- σ^2 exists → no fat tails

Related: Law of Large #'s

Let $X_1, X_2, \ldots$ be a sequence of random variables having a common distribution and let

$$E(X_i) = \mu$$

expectation

Then with probability 1;

$$\frac{X_1 + X_2 + \cdots X_n}{n} \to \mu$$

as $n \to \infty$

In simple terms:

$n \to \infty$, X_i: random variable

- LN: Average conv. to $E(X_i)$
- CLT: Sum Distribution $\Rightarrow$ Gaussian
  width $N\sigma^2$, centroid $N\mu$.